

A geometric application of Nori's connectivity theorem

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0 Introduction

Our purpose in this paper is to study rational maps from varieties with small dimensional moduli space to general hypersurfaces in projective space. In the last section, we shall eventually extend this to the study of correspondences instead of rational maps.

Definition 1 *Let $\mathcal{Y} \rightarrow \mathcal{S}$ be a family of r -dimensional smooth projective varieties. We say that a d -dimensional variety X is rationally swept out by varieties parametrized by \mathcal{S} , if there exist a quasiprojective variety B of dimension $d - r$, a family $K \rightarrow B$ which is the pull-back of the family \mathcal{Y} via a morphism $\psi : B \rightarrow \mathcal{S}$, and a dominant rational map*

$$\phi : K \dashrightarrow X,$$

(which is necessarily generically finite on the generic fiber K_b since $\dim K = \dim X$).

Our main result in this paper concerns the problem of sweeping out general hypersurfaces of degree $N \geq d + 2$ in \mathbb{P}^{d+1} :

Theorem 1 *Fix an integer $1 \leq r \leq d$. Let $\gamma = \frac{r-1}{2}$, r odd, or $\gamma = \frac{r}{2}$, r even, that is γ is the round-up of $\frac{r-1}{2}$. Let $\mathcal{Y} \rightarrow \mathcal{S}$, $\dim \mathcal{S} = C$, be a family of r -dimensional smooth projective varieties. Then the general hypersurface of degree N in \mathbb{P}^{d+1} is not rationally swept out by varieties parameterized by \mathcal{S} if*

$$(N + 1)r \geq 2d + C + 2, (\gamma + 1)N \geq 2d - r + 1 + C. \quad (0.1)$$

(Note that except for $r = 1$, the second inequality implies the first.)

Remark 1 *One could of course prove a similar statement for sufficiently ample hypersurfaces in any smooth variety. In the case of projective space, the estimates on N are sharp, and allow applications to the Calabi-Yau case (see section 3).*

The proof is Hodge theoretic. Unlike [4], [15], [7], [3], the result has nothing to do with the canonical bundle of the varieties Y_t , $t \in \mathcal{S}$. Instead, the key point is the fact that the dimension of the moduli space \mathcal{S} is small : if every X was rationally swept out by varieties parameterized by \mathcal{S} , for fixed Y , there would be a generically finite rational map

$$Y \times \tilde{U}_Y \dashrightarrow \mathcal{X}_{\tilde{U}_Y}$$

where $\mathcal{X}_{\tilde{U}_Y}$ is the pull-back via a morphism $\rho : \tilde{U}_Y \rightarrow U$ of the universal hypersurface parameterized by $U \subset H^0(\mathcal{O}_{\mathbb{P}^{d+1}}(N))$, and $\text{Im } \rho$ is of codimension $\leq \dim \mathcal{S}$. This will be shown to contradict Nori's connectivity theorem (Theorem 4).

We shall apply this particularly to the case of Calabi-Yau hypersurfaces. In the paper [11], Lang formulates a number of conjectures concerning smooth projective complex varieties X . One of them is that the analytic closure of the union of the images of holomorphic maps from \mathbb{C} to X is equal to the union of the images of non constant rational maps from an abelian variety to X . Another one is that this locus is equal to X itself if and only if X is not of general type.

Next, by a standard countability argument for Chow varieties, we see that, according to these conjectures, if X is not of general type, there should exist a quasiprojective variety B , a family $K \rightarrow B$ of abelian varieties, and a dominant rational map

$$\phi : K \dashrightarrow X,$$

which is non constant on the generic fiber K_b , $b \in B$.

Let us now consider the case where X is a Calabi-Yau variety, that is K_X is trivial. We claim that if a map ϕ as above exists, then we may assume that $\phi|_{K_b}$ is generically finite, for generic $b \in B$. Indeed, because $H^0(X, K_X) \neq 0$, for generic $b \in B$, the image $\phi(K_b)$ has effective canonical bundle, in the sense that any desingularization of it has effective canonical bundle, as follows from adjunction formula and the fact that the $\phi(K_b)$ cover X . Now it is immediate to prove that any dominant rational map

$$K_b \dashrightarrow K'_b,$$

where K_b is an abelian variety and K'_b has effective canonical bundle, factors through the quotient map $K_b \rightarrow K''_b$, where K''_b is an abelian variety, which is a quotient of K_b , and has the same dimension as K'_b . Replacing the family of abelian varieties $(K_b)_{b \in B}$, by the family $(K''_b)_{b \in B}$ gives the desired ϕ' .

In other words, Lang's conjecture asserts in particular that a Calabi-Yau variety should be rationally swept out by r -dimensional abelian varieties, for some $r \geq 1$. Our theorem 1 implies :

Theorem 2 *Let X be a general Calabi-Yau hypersurface in projective space \mathbb{P}^{d+1} , that is $N = d + 2$. Then X is not rationally swept out by r -dimensional abelian varieties, for any $r \geq 2$.*

Hence, if Lang's conjecture is true, such an X should be swept out by elliptic curves.

On the other hand, we also prove the following

Lemma 1 *If X is a general Calabi-Yau hypersurface of dimension ≥ 2 , X is not rationally swept out by elliptic curves of fixed modulus.*

By “rationally swept out by elliptic curves of fixed modulus”, we mean that the elliptic curves in the family $K \rightarrow B$ of definition 1 have constant modulus.

Hence, combining theorem 2 with the above lemma, we get the following corollary, which was pointed out to us by J. Harris :

Corollary 1 *If Lang's conjecture is true, any Calabi-Yau hypersurface X of dimension ≥ 2 has a divisor which is uniruled.*

In dimension 3, this shows that Lang's conjecture and Clemens conjecture on the finiteness of rational curves of fixed degree in a general quintic threefold, contradict.

In the case of hypersurfaces of general type, inequality (0.1) can be applied to give a non trivial estimate on the minimal genus of covering families of curves, but the estimate is not sharp and could be obtained directly by geometry. What is interesting however is that looking more precisely at the proof of Theorem 1, we shall see that the result concerns in fact only the Hodge structure on $H^d(X)_{prim}$ and not the effective geometry of X . In fact we get as well :

Theorem 3 *Let X be a general hypersurface of degree $N \geq 2d - 2 + 3g$, $g \geq 2$ or $N \geq 2d + 2$, $g = 1$, in \mathbb{P}^{d+1} . Then there exists no non-zero morphism of Hodge structure*

$$H^d(X, \mathbb{Q})_{prim} \rightarrow H^d(Y, \mathbb{Q}),$$

where Y is rationally swept out by curves of genus g .

Combining this statement with the generalization of Mumfords theorem [12], this implies in particular that for $N \geq 2d + 2$, X general, there exists no correspondence $\Gamma \in CH^d(Y \times X)$ inducing a surjective map

$$\Gamma_* : CH_0(Y)_0 \rightarrow CH_0(X)_0,$$

where Y admits an elliptic fibration. Similarly, if $g \geq 2$ and $N \geq 2d + 2$, there exists no such correspondence $\Gamma \in CH^d(Y \times X)$ where Y admits a fibration whose generic fiber is a genus g curve. One may wonder whether these statement are true for any such hypersurface or only for the general one.

The paper is organized as follows : in section 1, we recall briefly the proof of Nori's connectivity theorem for hypersurfaces in projective space, in order to extend it to families of hypersurfaces parameterized by subvarieties of the moduli space which are of small codimension. This will show us that for any family of hypersurfaces parameterized by a subvariety of the moduli space

which is of small codimension, the Hodge level of the cohomology groups of the total space of the family is small.

The next section is devoted to the proof (by contradiction) of theorem 1.

In section 3, we prove the applications of this result described above.

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1 Nori’s connectivity theorem for hypersurfaces

In this section, we summarize the main points of the proof of Nori’s connectivity theorem for hypersurfaces, in order to prove theorem 4, which is the precision of it that we will need. in [1], [14], a sharper study of similar explicit bounds can be found.

We consider hypersurfaces of degree N in \mathbb{P}^{d+1} , and we assume that $N \geq d + 2$. Fix an integer r such that $1 \leq r \leq d$, and let γ be the round-up of $\frac{r-1}{2}$. Denote by $U \subset H^0(\mathcal{O}_{\mathbb{P}^{d+1}}(N))$ the open set parametrizing smooth hypersurfaces. Let $\rho : \mathcal{M} \rightarrow U$ be a morphism, where \mathcal{M} is smooth quasi-projective. We assume that $\text{Corank } \rho$ is constant equal to C . We also assume for simplicity that $\text{Im } \rho$ is stable under the action of $Gl(N)$. Let \mathcal{X}_U be the universal hypersurface parametrized by U and

$$\mathcal{X}_{\mathcal{M}} := \mathcal{X}_U \times_U \mathcal{M}.$$

Let

$$j : \mathcal{X}_{\mathcal{M}} \hookrightarrow \mathcal{M} \times \mathbb{P}^{d+1}$$

be the natural embedding. $\mathcal{X}_{\mathcal{M}}$ is a smooth quasi-projective variety, hence its cohomology groups carry mixed Hodge structures with associated Hodge filtration $F^i H^k(\mathcal{X}_{\mathcal{M}}, \mathbb{C})$.

Theorem 4 *i) Assume that*

$$(N + 1)r \geq 2d + C + 2. \tag{1.2}$$

Then, the restriction map

$$j^* : F^d H^{2d-r}(\mathcal{M} \times \mathbb{P}^{d+1}, \mathbb{C}) \rightarrow F^d H^{2d-r}(\mathcal{X}_{\mathcal{M}}, \mathbb{C})$$

is surjective.

ii) If

$$(\gamma + 1)N \geq 2d + 1 - r + C, \quad (1.3)$$

then for any $i \geq 1$, the restriction map

$$j^* : H^{2d-r-i}(\mathcal{M} \times \mathbb{P}^{d+1}, \mathbb{C}) \rightarrow H^{2d-r-i}(\mathcal{X}_{\mathcal{M}}, \mathbb{C})$$

is surjective.

Proof. i) One first reduces i), (see [13]), to proving that under the assumption (1.2), the restriction map

$$j^* : H^l(\Omega_{\mathcal{M} \times \mathbb{P}^{d+1}}^k) \rightarrow H^l(\Omega_{\mathcal{X}_{\mathcal{M}}}^k)$$

is bijective, for $l \leq d - r$, $k + l \leq 2d - r$. This step uses the mixed Hodge structure on relative cohomology and the Fröhlicher spectral sequence.

Denote respectively by π_X , $\pi_{\mathbb{P}}$ the natural maps

$$\mathcal{X}_{\mathcal{M}} \rightarrow \mathcal{M}, \mathcal{M} \times \mathbb{P}^{d+1} \rightarrow \mathcal{M}.$$

A Leray spectral sequence argument shows that it suffices to prove that under the assumption (1.2) one has :

$$\text{The restriction map } j^* : R^l \pi_{\mathbb{P}*}(\Omega_{\mathcal{M} \times \mathbb{P}^{d+1}}^k) \rightarrow R^l \pi_{X*}(\Omega_{\mathcal{X}_{\mathcal{M}}}^k) \text{ is bijective} \quad (1.4)$$

for $l \leq d - r$, $k + l \leq 2d - r$.

Let

$$\mathcal{H}_{prim}^d, \mathcal{H}_{prim}^{p,q}, p + q = d,$$

be the Hodge bundles associated to the variation of Hodge structure on the primitive cohomology of the family $\pi_X : \mathcal{X}_{\mathcal{M}} \rightarrow \mathcal{M}$. The infinitesimal variation of Hodge structure on the primitive cohomology of the fibers of π_X is described by maps

$$\bar{\nabla} : \mathcal{H}_{prim}^{p,q} \rightarrow \mathcal{H}_{prim}^{p-1,q+1} \otimes \Omega_{\mathcal{M}},$$

and they can be iterated to produce a complex :

$$\dots \mathcal{H}_{prim}^{p+1,q-1} \otimes \Omega_{\mathcal{M}}^{s-1} \xrightarrow{\bar{\nabla}} \mathcal{H}_{prim}^{p,q} \otimes \Omega_{\mathcal{M}}^s \xrightarrow{\bar{\nabla}} \mathcal{H}_{prim}^{p-1,q+1} \otimes \Omega_{\mathcal{M}}^{s+1} \dots \quad (1.5)$$

One can show, using the filtration of $\Omega_{\mathcal{X}_{\mathcal{M}}}^k$ by the subbundles $\pi_X^* \Omega_B^s \wedge \Omega_{\mathcal{X}_{\mathcal{M}}}^{k-s}$ and the associated spectral sequence, that (1.4) is equivalent to the following

The sequence (1.5) is exact at the middle for $q \leq d - r$, $p + s + q \leq 2d - r$.

Note that since $p + q = d$, the last inequality reduces to $s \leq d - r$.

It is convenient to dualize (1.5) using Serre duality, which gives :

$$\mathcal{H}_{prim}^{q+1,p-1} \otimes \bigwedge^{s+1} T_{\mathcal{M}} \xrightarrow{t\bar{\nabla}} \mathcal{H}_{prim}^{q,p} \otimes \bigwedge^s T_{\mathcal{M}}^s \xrightarrow{t\bar{\nabla}} \mathcal{H}_{prim}^{q-1,p+1} \otimes \bigwedge^{s-1} T_{\mathcal{M}}^{s+1}. \quad (1.6)$$

We finally use Griffiths, Griffiths-Carlson description of the IVHS of hypersurfaces ([10], [2]) to describe the complex (1.6) at the point $f \in \mathcal{M}$ as follows. We have the map $\rho_* : T_{\mathcal{M},f} \rightarrow T_{U,f} = S^N$, where S is the polynomial ring in $d+2$ variables. Next the residue map provides isomorphisms

$$R_f^{-d-2+N(p+1)} \cong H_{prim}^{q,p}(X_f),$$

where $R_f := S/J_f$ is the Jacobian ideal of f , and R_f^k denotes its degree k component. The map $\bar{\nabla}$ identifies then, up to a coefficient, to the map given by multiplication

$$R_f^{-d-2+N(p+1)} \rightarrow \text{Hom}(T_{\mathcal{M},f}, R_f^{-d-2+N(p+2)}).$$

It follows from this that the sequence (1.6) identifies to the following piece of the Koszul complex of the Jacobian ring R_f with respect to the action of $T_{\mathcal{M},f}$ on it by multiplication:

$$R_f^{-d-2+Np} \otimes \bigwedge^{s+1} T_{\mathcal{M},f} \xrightarrow{\delta} R_f^{-d-2+N(p+1)} \otimes \bigwedge^s T_{\mathcal{M},f} \xrightarrow{\delta} R_f^{-d-2+N(p+2)} \otimes \bigwedge^{s-1} T_{\mathcal{M},f} \quad (1.7)$$

Now, by assumption, if W is the image of ρ_* , $W \subset S^N$ is a base-point free linear system, because it contains the jacobian ideal J_f^N , and it satisfies $\text{codim } W = C$.

One verifies that it suffices to check exactness at the middle of the exact sequences (1.7) in the considered range, with $T_{\mathcal{M},f}$ replaced with W . This last fact is then a consequence of the following theorem due to M. Green :

Theorem 5 [9] *Let $W \subset S^N$ be a base-point free linear system. Then the following sequence, where the differentials are the Koszul differentials*

$$S^{-d-2+Np} \otimes \bigwedge^{s+1} W \xrightarrow{\delta} S^{-d-2+N(p+1)} \otimes \bigwedge^s W \xrightarrow{\delta} S^{-d-2+N(p+2)} \otimes \bigwedge^{s-1} W \quad (1.8)$$

is exact for $-d-2+Np \geq s + \text{codim } W$.

Using the fact that the Jacobian ideal is generated by a regular sequence in degree $N-1$, one then shows that the same is true when S^i is replaced with R_f^i in (1.8), at least if $-d-2+N(p+1) \geq N-1$.

We now conclude the proof of i). We have just proved that (1.5) is exact at the middle if

$$-d-2+Np \geq s+C, \quad -d-2+N(p+1) \geq N-1.$$

Since we assumed $N \geq d+2$, the second inequality is satisfied when $p \geq 1$. Next, if $q \leq d-r$, $s \leq d-r$, we have

$$p \geq r \geq 1, \quad s \leq d-r.$$

Hence, the exactness of (1.5) in the range $q \leq d - r$, $s \leq d - r$ will follow from the inequality

$$-d - 2 + Nr \geq d - r + C,$$

that is (1.2).

ii) The proof is exactly similar, and we just sketch it in order to see where the numerical assumption is used. We first observe that by a mixed Hodge structure argument (cf [13]), it suffices, in order to get the surjectivity of the restriction map :

$$j^* : H^{2d-r-i}(\mathcal{M} \times \mathbb{P}^{d+1}, \mathbb{C}) \rightarrow H^{2d-r-i}(\mathcal{X}_{\mathcal{M}}, \mathbb{C}),$$

to show the surjectivity of the restriction map :

$$j^* : F^{d-r+\gamma_i} H^{2d-r-i}(\mathcal{M} \times \mathbb{P}^{d+1}, \mathbb{C}) \rightarrow F^{d-r+\gamma_i} H^{2d-r-i}(\mathcal{X}_{\mathcal{M}}, \mathbb{C}),$$

where γ_i is the round-up of $\frac{r-i}{2}$. (This is because the round-up of $\frac{2d-r-i}{2}$ is $d - r + \gamma_i$.) We reduce then this last fact to showing :

$$\text{The restriction map } j^* : R^l \pi_{\mathbb{P}*}(\Omega_{\mathcal{M} \times \mathbb{P}^{d+1}}^k) \rightarrow R^l \pi_{X*}(\Omega_{\mathcal{X}_{\mathcal{M}}}^k) \text{ is bijective} \quad (1.9)$$

for $l \leq d - i - \gamma_i$, $k + l \leq 2d - r - i$.

Expressing the cohomology groups above with the help of the IVHS on the primitive cohomology of the fibers of π_X , this is reduced to proving :

The sequence (1.5) is exact at the middle for $q \leq d - i - \gamma_i$, $p + q = d$, $p + s + q \leq 2d - r - i$.

Using the Carlson-Griffiths theory, we are now reduced to prove :

the following sequence :

$$R_f^{-d-2+Np} \otimes \bigwedge^{s+1} T_{\mathcal{M},f} \xrightarrow{\delta} R_f^{-d-2+N(p+1)} \otimes \bigwedge^s T_{\mathcal{M},f} \xrightarrow{\delta} R_f^{-d-2+N(p+2)} \otimes \bigwedge^{s-1} T_{\mathcal{M},f} \quad (1.10)$$

is exact for $p \geq \gamma_i + i$, $s \leq d - r - i$.

As in the previous proof, we now apply the theorem 5 and conclude that the last statement is true if

$$-d - 2 + N(\gamma_i + i) \geq C + d - r - i. \quad (1.11)$$

Now it is clear that the $\gamma_i + i$ are increasing with i , while the $C + d - r - i$ are decreasing with i . Hence it suffices to have (1.11) satisfied for $i = 1$, which is exactly inequality (1.3). \blacksquare

Denoting by $H^{2d-r}(\mathcal{X}_{\mathcal{M}})_{prim}$ the quotient

$$H^{2d-r}(\mathcal{X}_{\mathcal{M}})_{prim}/j^*(H^{2d-r}(\mathcal{M} \times \mathbb{P}^{d+1})),$$

we shall only be interested with the pure part

$$W_{2d-r}H^{2d-r}(\mathcal{X}_{\mathcal{M}})_{prim},$$

which is the part of the cohomology which comes from any smooth projective compactification of $\mathcal{X}_{\mathcal{M}}$. It carries a pure Hodge structure of weight $2d - r$.

Corollary 2 *Under the assumptions of theorem 4, the Hodge structure on $W_{2d-r}H^{2d-r}(\mathcal{X}_{\mathcal{M}})_{prim}$ is of Hodge level $\leq r - 2$.*

Proof. Recall that the Hodge level of a Hodge structure H , $H_{\mathbb{C}} = \oplus H^{p,q}$ is

$$\text{Max} \{p - q, H^{p,q} \neq 0\}.$$

Since we know that $F^d W_{2d-r}H^{2d-r}(\mathcal{X}_{\mathcal{M}})_{prim} = 0$, we have

$$H^{p,q}(W_{2d-r}H^{2d-r}(\mathcal{X}_{\mathcal{M}})_{prim}) = 0, \text{ for } p \geq d.$$

Since $H^{p,q} = 0$ for $p + q \neq 2d - r$, it follows that the Hodge level is $\leq d - 1 - (2d - r - (d - 1)) = r - 2$. \blacksquare

2 Proof of theorem 1

We prove theorem 1 by contradiction. Using Chow varieties, or relative Hilbert schemes, we see that there exist countably many quasi-projective varieties \mathcal{B} parameterizing triples (t, f, ϕ_s) , where $t \in \mathcal{S}$, $f \in U$, and ϕ_s is a rational map $\phi_s : Y_s \dashrightarrow X_f$ which is generically finite onto its image. For fixed generic f , our assumption is that the images of such ϕ_s fill-in X_f , and a countability argument then shows that there exists one \mathcal{B} , which dominates U via the second projection, and which is such that the universal rational map

$$\Phi : \mathcal{K} \dashrightarrow \mathcal{X}_U$$

is dominating, where

$$\pi : \mathcal{K} \rightarrow \mathcal{B}$$

is the pull-back via the first projection $\Psi : \mathcal{B} \rightarrow \mathcal{S}$ of the family $\mathcal{Y} \rightarrow \mathcal{S}$, and, as in the previous section, \mathcal{X}_U is the universal hypersurface parameterized by U . We shall denote by B_f the (generic) fiber of the second projection $q : \mathcal{B} \rightarrow U$ and $\pi_f : K_f \rightarrow B_f$ the induced family. By taking desingularizations, we may assume that \mathcal{B} hence \mathcal{K} are smooth, and by assumption the map π is smooth. Since f is generic, B_f and K_f are then also smooth. Finally, we may, up to replacing \mathcal{B} by a closed subvariety, assume that the restriction $\phi_f : K_f \dashrightarrow X_f$ of Φ to K_f is generically finite and dominating. In particular, $\dim B_f = d - r$.

Now we make the following construction : denote by \mathcal{B}_f the space

$$\mathcal{B} \times_{\mathcal{S}} B_f.$$

Restricting to Zariski open sets of \mathcal{B} and B_f , we may assume that \mathcal{B}_f is smooth. The generic point of \mathcal{B}_f parameterizes, via the second projection $p : \mathcal{B}_f \rightarrow B_f$, a variety Y_t together with a rational map $\phi_{t,f} : Y_t \dashrightarrow X_f$, and, via the second projection, a rational map

$$\phi_{t,f'} : Y_t \dashrightarrow X_{f'}.$$

Let $\rho : \mathcal{B}_f \rightarrow U$ be the composition of the first projection and the map $q : \mathcal{B} \rightarrow U$ and let $m : \mathcal{B}_f \rightarrow \mathcal{S}$ be the natural map. We shall also use the notation

$$\mathcal{K}_f = \mathcal{K} \times_{\mathcal{B}} \mathcal{B}_f = \bigcup_{t \in \mathcal{B}_f} Y_{m(t)},$$

and

$$\mathcal{X}_f := \mathcal{X} \times_U \mathcal{B}_f = \bigcup_{t \in \mathcal{B}_f} X_{\rho(t)}.$$

The map Φ induces a rational map $\Phi_f : \mathcal{K}_f \dashrightarrow \mathcal{X}_f$ which is compatible with the maps $\mathcal{K}_f \rightarrow \mathcal{B}_f$ and $\mathcal{X}_f \rightarrow \mathcal{B}_f$. It follows that the graph Γ of Φ is contained in

$$\mathcal{Y}_f := \mathcal{K}_f \times_{\mathcal{B}_f} \mathcal{X}_f \cong K_f \times_{B_f} \mathcal{X}_f$$

and is of codimension d in \mathcal{Y}_f .

Note that \mathcal{Y}_f contains $K_f \times X_f$ and that $\Gamma \cap K_f \times X_f$ is nothing but the graph of ϕ_f . Now, the class of this last graph in $H^{2d}(K_f \times X_f, \mathbb{Q})$ does not vanish in

$$H^{2d}(K_f \times X_f, \mathbb{Q}) / H^{2d}(K_f \times \mathbb{P}^{d+1}, \mathbb{Q}).$$

Indeed, its Künneth component ϕ_f^* in

$$\text{Hom}(H^d(X, \mathbb{Q})_{\text{prim}}, H^d(K_f, \mathbb{Q}))$$

does not vanish, because $N \geq d+2$, so that the transcendent part of $H^d(X_f, \mathbb{Q})_{\text{prim}}$, that is the orthogonal of all sub-Hodge structures which are of level $< d$, is non-zero, so that it cannot be annihilated by ϕ_f^* , because ϕ_f is dominating.

It follows that the class γ of Γ in $H^{2d}(\mathcal{Y}_f, \mathbb{Q})$ is non zero modulo $H^{2d}(\mathcal{K}_f \times \mathbb{P}^{d+1}, \mathbb{Q})$.

Recall next that with the help of a polarization, that is a choice of a relatively ample line bundle on $\mathcal{Y}_f \xrightarrow{\pi} \mathcal{X}_f$, the cohomology $H^{2d}(\mathcal{Y}_f, \mathbb{Q})$ splits into a direct sum

$$H^{2d}(\mathcal{Y}_f, \mathbb{Q}) = \bigoplus_l H^{2d-l}(\mathcal{X}_f, R^l \pi_* \mathbb{Q}).$$

It is easy to check by similar reasons as above that the component γ_r of γ in

$$H^{2d-r}(\mathcal{X}_f, R^r \pi_* \mathbb{Q}) / H^{2d-r}(\mathcal{B}_f \times \mathbb{P}^{d+1}, R^r \pi'_* \mathbb{Q}),$$

where in the second term π' is the natural map

$$\mathcal{K}_f \times \mathbb{P}^{d+1} \rightarrow \mathcal{B}_f \times \mathbb{P}^{d+1},$$

is non zero. We may finally for the same reason refine this, replacing $R^r \pi_* \mathbb{Q}$ by its quotient $R^r \pi_* \mathbb{Q}_{tr}$, that is its quotient by the maximal sub-Hodge structure which exists generically on B_f and is not of maximal Hodge level r . Note that γ , γ_r and $\gamma_{r,tr}$ are Hodge classes, that is belong to the $F^d W_{2d}$ -level of the considered cohomology groups, which all have mixed Hodge structures.

Next we denote by

$$g : \mathcal{B}_f \rightarrow B_f, \tilde{g} := g \circ \pi_X : \mathcal{X}_f \rightarrow B_f,$$

the natural maps. We observe that shrinking B_f if necessary, the fibers $\mathcal{B}_{f,t}$ of g are smooth and the map $\rho|_{\mathcal{B}_{f,t}}$ has constant corank $\geq C = \dim \mathcal{S}$, because $\rho : \mathcal{B} \rightarrow U$ is submersive near B_f , and the fibers of g identify to the fibers of $\mathcal{B} \rightarrow \mathcal{S}$.

Hence we may apply theorem 4 and its corollary. It says that under the assumptions 0.1, the fibers $\mathcal{X}_{f,t}$ of \tilde{g} satisfy $H^{2d-r-i}(\mathcal{X}_{f,t}, \mathbb{Q}) = 0$, $i \geq 1$.

On the other hand, we observe that because $\pi : \mathcal{Y}_f \rightarrow \mathcal{X}_f$ is a fiber product $K_f \times_{B_f} \mathcal{X}_f$, the local system $R^r \pi_* \mathbb{Q}$ is a pull-back from B_f :

$$R^r \pi_* \mathbb{Q} \cong \tilde{g}^{-1}(R^r \pi_{f*} \mathbb{Q}).$$

The vanishing above implies then that

$$(R^{2d-r-i} g_*(R^r \pi_* \mathbb{Q}_{tr}))_{prim} = 0, i \geq 1,$$

where “*prim*” here denotes the quotient by the ambient cohomology (while before it was used to mean the orthogonal to ambient cohomology). It follows by Leray spectral sequence that the class $\gamma_{r,tr}$ does not vanish along the fibers of g , that is in $H^0(R^{2d-r} g_*(R^r \pi_* \mathbb{Q}_{tr}))$ and by restriction to the general fiber, it does not vanish in

$$H^r(Y_t, \mathbb{Q})_{tr} \otimes H^{2d-r}(\mathcal{X}_{f,t}, \mathbb{Q}) / H^{2d-r}(\mathcal{B}_{f,t} \times \mathbb{P}^{d+1}, \mathbb{Q}).$$

Note that it is a Hodge class in

$$\begin{aligned} & H^r(Y_t, \mathbb{Q})_{tr} \otimes H^{2d-r}(\mathcal{X}_{f,t}, \mathbb{Q}) / H^{2d-r}(\mathcal{B}_{f,t} \times \mathbb{P}^{d+1}, \mathbb{Q}) \\ & \cong \text{Hom}(H^r(Y_t, \mathbb{Q})_{tr}, H^{2d-r}(\mathcal{X}_{f,t}, \mathbb{Q}) / H^{2d-r}(\mathcal{B}_{f,t} \times \mathbb{P}^{d+1}, \mathbb{Q})). \end{aligned}$$

Since the mixed Hodge structure on the left is pure, this implies that it belongs to

$$\begin{aligned} & \text{Hom}_{MHS}(H^r(Y_t, \mathbb{Q})_{tr}, H^{2d-r}(\mathcal{X}_{f,t}, \mathbb{Q}) / H^{2d-r}(\mathcal{B}_{f,t} \times \mathbb{P}^{d+1}, \mathbb{Q})) \\ & \cong \text{Hom}_{HS}(H^r(Y_t, \mathbb{Q})_{tr}, W_{2d-r} H^{2d-r}(\mathcal{X}_{f,t}, \mathbb{Q}) / H^{2d-r}(\mathcal{B}_{f,t} \times \mathbb{P}^{d+1}, \mathbb{Q})), \end{aligned}$$

where “MHS” and “HS” mean morphisms of mixed (respectively pure) Hodge structures.

Since by definition $H^r(Y_t, \mathbb{Q})_{tr}$ has no quotient Hodge structure which is of Hodge level $< r$, we get now a contradiction with corollary 2, which says that the Hodge structure on $W_{2d-r} H^{2d-r}(\mathcal{X}_{f,t}, \mathbb{Q}) / H^{2d-r}(\mathcal{B}_{f,t} \times \mathbb{P}^{d+1}, \mathbb{Q})$ has Hodge level $< r$. ■

3 Rational maps from abelian varieties to Calabi-Yau hypersurfaces and other applications

Let us apply theorem 1 to the case of Calabi-Yau hypersurfaces, that is hypersurfaces of degree $N = d + 2$ in \mathbb{P}^{d+1} . The moduli space of r -dimensional abelian varieties with given polarization type is of dimension $\frac{r(r+1)}{2}$. Hence the conditions of theorem 1 become :

$$(d+3)r \geq 2d + \frac{r(r+1)}{2} + 2, (\gamma+1)(d+2) \geq 2d - r + 1 + \frac{r(r+1)}{2}$$

. It is not hard to check that this is satisfied for $2 \leq r \leq d$. Hence we get in this case theorem 2. \blacksquare

When $r = 1$, the inequality (0.1) is never satisfied so that our argument definitely does not apply to the study of elliptic curves in Calabi-Yau hypersurfaces. In fact we could adapt our proof of theorem 2 to work as well for Calabi-Yau hypersurfaces in a product of projective spaces. On the other hand, certain generic Calabi-Yau hypersurfaces in a product of projective spaces are swept out by elliptic curves, eg the hypersurface of bidegree (3, 3) in $\mathbb{P}^2 \times \mathbb{P}^2$. This shows that for $r = 1$, a different argument has to be found.

We can however prove the following :

Lemma 2 *If X is a Calabi-Yau hypersurface of dimension ≥ 2 , X is not rationally swept out by elliptic curves of fixed modulus.*

Proof. Indeed, fixing otherwise the modulus of the elliptic curve, we would get, for at least one elliptic curve E , an hypersurface \mathcal{M}_E in the moduli space \mathcal{M} of X consisting of X_f 's which are rationally dominated by some $E \times B$. For such an X_f , there must be an inclusion of rational Hodge structures induced by the dominant rational map $\phi : E \times B \dashrightarrow X_f$:

$$\phi^* : H^d(X_f)_{\text{prim}} \hookrightarrow H^1(E) \otimes H^{d-1}(B), \quad (3.12)$$

because the Hodge structure on $H^d(X)_{\text{prim}}$ is simple.

If we now let f vary in \mathcal{M}_E , only B deforms with f , not E , and it follows that the infinitesimal variation of Hodge structure on $H^1(E) \otimes H^{d-1}(B)$

$$\overline{\nabla} : H^{p,q}(H^1(E) \otimes H^{d-1}(B)) \rightarrow H^{p-1,q+1}(H^1(E) \otimes H^{d-1}(B)) \otimes \Omega_{\mathcal{M}_E}$$

has the following form at the point $f \in \mathcal{M}_E$:

$$\overline{\nabla}(\alpha \otimes \beta) = \alpha \otimes \overline{\nabla}_B(\beta)$$

for $\alpha \in H^{r,s}(E)$, $\beta \in H^{p-r,q-s}(B)$, where $\overline{\nabla}_B$ is the infinitesimal variation of Hodge structure on $H^{d-1}(B)$. Hence the Yukawa couplings of the IVHS on $H^1(E) \otimes H^{d-1}(B)$, that is the iterations of $\overline{\nabla}$, have the following property:

$\forall \eta \in H^{d,0}(H^1(E) \otimes H^{d-1}(B))$, the map

$$\bar{\nabla}^d(\eta) : S^d T_{\mathcal{M}_E, f} \rightarrow H^{0,d}(H^1(E) \otimes H^{d-1}(B))$$

vanishes.

If there is along \mathcal{M}_E an injective morphism of Hodge structure (3.12), it follows that the same property is true for the IVHS of the family of X_f 's parameterized by \mathcal{M}_E , namely the Yukawa couplings of X_f vanish on the hyperplane $K := T_{\mathcal{M}_E, f} \subset S^{d+2}$. The Carlson-Griffiths theory [10], [2] shows easily that this is not the case. Indeed, these Yukawa couplings identify to the multiplication map

$$S^d(S^{d+2}) \rightarrow R_f^{d(d+2)}.$$

Assume they vanish on K . Since K is a hyperplane in S^{d+2} , the subspace

$$K' := [K : S^1] \subset S^{d+1}$$

has codimension $\leq d+2$. It is without base-point, since $T_{\mathcal{M}_E, f}$ contains J_f . It follows then from [8], that $S^{d+3}K' = S^{2d+4}$. But K^2 contains $S^1K' \cdot K = K' \cdot S^1K = W \cdot S^{d+3} = S^{2d+4}$. Hence $K^2 = S^{2d+4}$ and similarly $K^d = S^{d(d+2)}$ contradicting the fact that $K^d \subset J_f^{d(d+2)}$. \blacksquare

Remark 2 *In the case of odd dimensional varieties, one can also use the Mumford-Tate group argument due to Deligne ([6], p 224) to get this result.*

Corollary 3 *If Lang's conjecture is true, any Calabi-Yau hypersurface X of dimension ≥ 2 has a divisor which is uniruled.*

Proof. Indeed, we know that X is rationally swept out by elliptic curves, but not by elliptic curves with constant modulus. Hence there is a diagram

$$\begin{array}{ccc} \widetilde{\mathcal{K}} & \xrightarrow{\phi} & X \\ \pi \downarrow & & \\ \widetilde{B} & & \end{array},$$

where we may assume that $\widetilde{\mathcal{K}}$ and \widetilde{B} are smooth, projective, where $\widetilde{\mathcal{K}}$ is a smooth projective model of the family $\mathcal{K} \rightarrow B$ on which ϕ is defined, and that the map $j : \widetilde{B} \rightarrow \mathbb{P}^1$ is defined. Now, since ϕ is generically finite, for generic $t \in \mathbb{P}^1$, the divisor $\widetilde{\mathcal{K}}_t := (j \circ \pi)^{-1}(t)$ must be sent by ϕ onto a divisor of X , and it follows that for any t the image by ϕ of the divisor $\widetilde{\mathcal{K}}_t := (j \circ \pi)^{-1}(t)$ must contain a divisor of X . Taking $t = \infty$, and noting that any component of $\widetilde{\mathcal{K}}_\infty$ is uniruled, gives the result. \blacksquare

Finally, we turn to Theorem 3. We simply note for this that in the proof of Theorem 1, we used the dominating rational map

$$\phi : K \dashrightarrow X$$

only to deduce that there is a corresponding inclusion

$$\phi_f^* : H^d(X_f)_{\text{prim}} \rightarrow H^d(K_f)$$

of Hodge structures. The Chow argument we used would work equally with graphs of rational maps replaced with cycles in the product

$$Y_t \times X.$$

Hence we conclude that everywhere in the paper, we could replace “dominating rational maps” by “cycles in $CH^d(K \times X)$ inducing a non-zero morphism of Hodge structure”

$$H^d(X)_{\text{prim}} \rightarrow H^d(K).$$

(Note that such a non-zero morphism should be in fact injective because the Hodge structure on $H^d(X)_{\text{prim}}$ is simple for generic X .) Using the fact that the moduli space of curves of genus $g \geq 1$ has dimension $3g - 3$, $g \geq 2$ or 1 , $g = 1$, we see that theorem 3 is then a consequence of theorem 1, where we replace dominating rational maps with correspondences inducing a non zero morphism of Hodge structure. ■

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